Weierstrass' theorem is the basis for Chapter 4. First the holomorphic version of Nersesyan (1971) is given, and then the related results for harmonic approximation follow. Chapter 5 deals with tangential approximation at infinity, where one tries to control the behavior of the error of approximation at infinity. This chapter complements the results from Chapter 3. Up to here the approximations are always in terms of harmonic functions. In Chapter 6 it is shown that many of the obtained results can be extended to superharmonic functions. Finally, the last two chapters contain some applications, such as a complete classification of all (unbounded) open sets $\Omega$ for which the Dirichlet problem can be solved (Chapter 7) and the existence of a nonconstant harmonic function for which the Radon transform vanishes and of other peculiar (pathological) harmonic functions (Chapter 8).

The author succeeded very well in writing a comprehensive book: the results are well motivated and illustrated (using the more commonly known results in holomorphic approximation), and at the end the author obtains some surprising applications. The reader is expected to have knowledge of complex analysis (with a background in subharmonic functions). The intended readership should consist of researchers in analysis, especially young researchers preparing a Ph.D. in analysis, but also postgraduates and more advanced researchers.

Walter Van Assche
E-mail: `walter@wis.kuleuven.ac.be`

P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics **161**, Springer-Verlag, New York/Berlin, 1995, x + 480 pp.

Polynomials are surely among the simplest functions that one deals with. They are easy to differentiate or to integrate, the fundamental theorem of algebra tells us that any polynomial of degree $n$ has precisely $n$ complex zeros, and they have a very simple analytic description as a finite Taylor series. These are some of the reasons approximators like to use polynomials for approximation of functions. Polynomial approximation moreover works quite well for continuous functions on a compact set of the real line (Weierstrass' theorem). As the authors point out in their preface, virtually every branch of mathematics has its corpus of theory arising from the study of polynomials. Hence polynomials really deserve special attention and a book devoted especially to polynomials will be of interest to many people interested in mathematics.

This book by Borwein and Erdélyi gives an interesting approach to polynomials, with emphasis in the later chapters on polynomial inequalities. Only polynomials in one variable are treated, so those interested in polynomials in several variables will be disappointed. But then the authors are using the word polynomial in a very broad sense and very quickly put most of the exposition in terms of an extended notion of polynomials. To them, a Chebyshev system, a Markov system, or a Descartes system contains most of the essentials of polynomials. Müntz polynomials receive a lot of attention, as do rational functions with prescribed poles. The authors' treatment of Müntz polynomials is impressive and probably this subject is not treated in such detail elsewhere.

The presentation of the book is unusual. Often the text gives an introduction to a particular aspect of the theory and then gives much more in the form of exercises. As an example, the Stone–Weierstrass theorem is given as exercise E.2 on p. 161, and the interested reader is led through the proof in seven steps by means of useful hints. This approach of presenting relevant results in such a way that the reader needs to work out the proof by himself is used throughout the book. The book is very similar in spirit to one of my favorite classics, *Problems and Theorems in Analysis*, by Pólya and Szegő. This presentation is very appealing and makes the book very attractive for classroom use. Parts of the book can be used in a graduate course, other more advanced topics can be used in a postgraduate course, and some of the material given in the appendices can be used for advanced seminars.

Chapter 1 is an introduction and gives basic properties of polynomials. Here we find the fundamental theorem of algebra (with the explicit solutions of quadratic, cubic, and quartic equations); interpolation formulas of Lagrange, Hermite, and Newton; location of zeros of polynomials (Eneström–Kakeya theorem); and zeros of the derivative of polynomials (theorems of Lucas, Jensen, and Walsh). Chapter 2 deals with special polynomials. Chebyshev polynomials of the first kind $T_n$ are introduced here as extremal polynomials in $L^\infty[-1, 1]$. These are fundamental polynomials that play a role in various problems regarding polynomials, in particular in polynomial inequalities. Chebyshev polynomials of the second kind are also given and for both systems their various properties are listed (orthogonality, recursion relation, explicit representation, composition properties, etc.). The notion of extremal polynomial is then extended to other compact sets of the complex plane by means of the transfinite diameter (capacity) and the Chebyshev constant. Then extremal polynomials in $L^2$ (orthogonal polynomials) are treated, with the classical properties of their zeros (real and simple, interlacing properties), the three-term recursion relation, and a short visit to the classical orthogonal polynomials (Jacobi, Hermite, Laguerre). Favard's theorem is proved, and some recent results on the relation between the recursion coefficients and the support of the orthogonality measure are mentioned. Other special polynomials treated in Chapter 2 are polynomials with nonnegative coefficients and positive polynomials on $(-1, 1)$, with their Bernstein representation and the corresponding Lorentz degree. Then we move to Chapter 3, where the authors fully explore their extended version of polynomials, namely Chebyshev and Descartes systems. They support their view by pointing out that from an approximator's point of view an essential property of polynomials of degree at most $n$ is that they can uniquely interpolate at $n+1$ points. But this is also true for any finite $(n+1)$-dimensional Chebyshev space (or Haar space). Chebyshev's alternation theorem for best approximants is proved, and then various examples of Chebyshev systems are explored in the exercises. Markov systems and Descartes systems are then introduced, together with the generalized notion of Chebyshev polynomials in such spaces. Müntz polynomials and rational systems get special attention because these systems can be analyzed in more detail. Müntz–Legendre polynomials, obtained by orthogonalizing $x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, ...$, are examined in some detail. Later they turn out to be quite useful not only in proving inequalities in Müntz spaces (Chapter 6) but also in proving irrationality of some real numbers (Appendix A2). The chapter ends with Chebyshev polynomials in rational spaces. Chapter 4 is a real treat for analysts. It treats denseness questions, starting with Weierstrass' theorem and some of its variations. Various proofs are explored, some using Bernstein polynomials, Korovkin's theorem, Fejér's theorem, or the behavior of the maximal gap between two consecutive zeros of Chebyshev polynomials. Müntz's theorem on the density of $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, ...\}$ in $C[0, 1]$ gets a very clear treatment, together with its variations in $L^2[0, 1]$ and $L^p[0, 1]$. Extensions to an interval $[a, b]$ with $a > 0$, and also with $a < 0 < b$, as well as nondense Müntz spaces get sufficient attention. For nondense Müntz polynomials one can still go to dense systems in $C[a, b]$, with $a \geqslant 0$, by considering ratios of Müntz polynomials. The beautiful result of Somorjai (conjectured by Newman) that such Müntz rationals, derived from any infinite Müntz system, are always dense in $C[a, b]$, is covered in detail. Finally, a characterization of denseness of Markov spaces is given in terms of the existence of an unbounded Bernstein inequality, thus moving to the polynomial inequalities.

Chapter 5 is completely devoted to the classical inequalities for algebraic and trigonometric polynomials, starting with the Remez inequality and of course including Bernstein, Markov, and Schur inequalities. There are, unfortunately, only two pages of motivation for studying such inequalities, i.e., pp. 241–242, where Bernstein's inequality is used to derive an inverse theorem of approximation. More applications such as this one would have been welcome. On the other hand, a great variety of pretty inequalities are offered: the Bernstein–Szegő inequality, Videnskii's inequalities, inequalities for entire functions of exponential type, Markov's inequality for higher derivatives, and weighted versions of the Bernstein and Markov inequalities. These inequalities are extended to Müntz spaces in Chapter 6, where Newman's inequality is the basic result, and to rational function spaces in Chapter 7.

At the end there are five appendices dealing with some additional material such as algorithms and computational complexity for polynomials and rational functions (Appendix A1); orthogonality and irrationality, with a proof of the irrationality of $\zeta(3)$, $\pi^2$, and $\log 2$ (Appendix A2); an interpolation theorem for linear functionals (Appendix A3); recent material on inequalities for generalized polynomials in $L^p$ (Appendix A4); and inequalities for polynomials with constraints (Appendix A5).

This is a wonderful book which is strongly recommended for use in a class with students who are willing to work on the proofs, rather than to digest fully prepared and worked out proofs and examples. I have already used Appendix A2 successfully in a few research seminars and believe that the material in the book and the approach taken by the authors will prove to be a success.

Walter Van Assche
E-mail: walter@wis.kuleuven.ac.be

K. B. Datta and B. M. Mohan, *Orthogonal Functions in Systems and Control*, Advanced Series in Electrical and Computer Engineering **9**, World Scientific, Singapore, 1995, xi + 275 pp.

Although a part of this book deals with the sine and cosine functions and with piecewise linear functions such as Haar, Rademacher, Walsh, and block pulses, the main emphasis is on traditional orthogonal polynomials from Legendre to Gegenbauer.

The fundamental role played by such orthogonal function systems in approximation theory and numerical analysis needs no advertising. They have a history going back to the 18th and 19th centuries. Their use in signal processing, differential equations, systems, and control is somewhat less widespread and much more recent. The early applications appear as late as the 1970s. The first chapter of this book is mainly an extensively annotated bibliography about the role these orthogonal systems played in those applications from about 1970 to 1990.

The second chapter gives a collection of all the basic properties of the orthogonal functions. The next chapter moves towards signal processing by discussing shifted versions of the polynomials, their approximating properties in the presence of noise, treatment of two-dimensional signals, and the effect of differentiation and integration. The latter leads to quadrature formulas by specifying how the integral of the basis functions is expressed in terms of the same set of basis functions.

Thus the actual part dealing with systems and control starts about halfway through the book with Chapter 4. In time-delay systems the state vector $x$ and the control vector $u$ are coupled by $\dot{x}(t) = Ax(t) + Bu(t) + Fx(t-\tau) + Gu(t-\tau)$, where $\tau$ is the delay, and $A$, $B$, $G$ and $F$ are matrices. For the solution of such systems, a framework of integration and delay integration is developed along the lines of the previous chapter.

The next three chapters deal with identification problems, i.e., to estimate parameters and possibly initial conditions and boundary conditions for a system. One has to solve a differential equation and the basic mathematical problem is to approximate repeated integration in "one shot" by a linear combination of the functions in the orthogonal system. The last chapter solves a control problem. That is, the optimal control vector $u$ is written as $u(t) = K(t)\,x(t)$, where $x$ is the state vector and $K(t)$ the gain matrix to be computed.